Subspaces in Linear Algebra: Solution Space, Column Space, Row Space, and Null Space

Introduction

In linear algebra, matrices define relationships between vectors and spaces. Understanding subspaces like the **solution space**, **column space**, **row space**, and **null space** helps us analyze these relationships. Below is an explanation of these concepts with examples.

1. Solution Space

The solution space of a matrix A corresponds to all possible solutions $\mathbf{x}$ that satisfy the system of linear equations $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b}$ is in the column space of A.

Example:

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, solve $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

- If $\mathbf{b} = \mathbf{0}$, the solution space coincides with the **null space.**
- If $\mathbf{b} \neq \mathbf{0}$, the solution space is a translated version of the null space.

2. Column Space

The column space of a matrix A (also called the range or image) is the span of its column vectors. It consists of all linear combinations of the columns of A, representing all possible outputs $\mathbf{b}$ for $A\mathbf{x} = \mathbf{b}$.

Properties:

- The column space is a subspace of $\mathbb{R}^m$, where m is the number of rows in A.
- Its dimension is the **rank** of A.

Example:

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$, the column space is spanned by:

$$\mathrm{Col}(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \right).$$

3. Row Space

The row space of A is the span of its row vectors. It represents all linear combinations of the rows of A.

Properties:

- The row space is a subspace of $\mathbb{R}^n$, where n is the number of columns in A.
- Its dimension equals the rank of A.

Example:

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, the row space is spanned by:

$$\mathrm{Row}(A) = \mathrm{span}\left(\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \right).$$

4. Null Space

The null space of A (also called the kernel) is the set of all vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. It measures the "degree of freedom" in the solutions of $A\mathbf{x} = \mathbf{b}$.

Properties:

- The null space is a subspace of $\mathbb{R}^n$, where n is the number of columns in A.
- Its dimension is the **nullity** of A, given by the formula:

$$\mathrm{nullity}(A) = n - \mathrm{rank}(A).$$

Example:

For $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, solve $A\mathbf{x} = \mathbf{0}$:

$$1x_1 + 2x_2 + 3x_3 = 0,$$
$$4x_1 + 5x_2 + 6x_3 = 0.$$

This gives:

$$\mathrm{Null}(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right).$$

5. Dimensional Relationships: Rank-Nullity Theorem

The Rank-Nullity Theorem connects the dimensions of these spaces:

$$\text{Rank}(A) + \text{Nullity}(A) = n,$$

where n is the number of columns in A.

Summary

- **Column Space**: Represents all possible outputs of $A\mathbf{x}$ (spans columns of A).
- **Row Space**: Represents constraints or equations $A\mathbf{x}$ must satisfy (spans rows of A).
- **Null Space**: Represents the set of all solutions to $A\mathbf{x} = 0$.
- **Solution Space**: Combines the null space with particular solutions for $A\mathbf{x} = \mathbf{b}$.