Finding the Basis of a Matrix

Introduction

The basis of a matrix refers to a set of linearly independent vectors that span a specific subspace associated with the matrix. This can include the column space, row space, or null space. Below, we explain how to find these bases, step-by-step, with detailed examples.

1. Basis for the Column Space (Column Basis)

The column space of a matrix A consists of all linear combinations of its columns. To find the column basis:

1. Write down the columns of the matrix $A = [\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n]$.
2. Perform Gaussian elimination to row-reduce A into reduced row echelon form (RREF).
3. Identify the pivot columns in the RREF. The corresponding columns of the original matrix A form the basis for the column space.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Performing row reduction, we find:

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The pivot columns are the 1st and 2nd columns. Hence, the column basis is:

$$\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} \right\}.$$

2. Basis for the Row Space (Row Basis)

The row space of A is the span of its rows. To find the row basis:

1. Perform Gaussian elimination to row-reduce A into RREF.
2. The non-zero rows in the RREF form the basis for the row space.

Example: Using the same matrix A:

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The non-zero rows are:

$$\{\begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}\}.$$

Thus, this set forms the row basis.

3. Basis for the Null Space (Null Basis)

The null space of A consists of all vectors $\mathbf{x}$ such that $A\mathbf{x} = 0$. To find the null basis:

1. Solve $A\mathbf{x} = 0$.
2. Write the augmented matrix and row-reduce it.
3. Express the solution vector in terms of free variables.
4. Extract the basis vectors corresponding to the free variables.

Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

Solve $A\mathbf{x} = 0$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Writing this as a system:

$$1x_1 + 2x_2 + 3x_3 = 0,$$
$$4x_1 + 5x_2 + 6x_3 = 0.$$

Write the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \end{array}\right].$$

Perform row reduction:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right].$$

From the reduced system:

$$x_1 = x_3, \quad x_2 = -2x_3.$$

Let $x_3 = t$, where t is a free parameter. The solution is:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix}$$

$$\mathbf{x} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

The null basis is:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

4. Additional Example for Null Space

Consider the matrix:

$$B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -2 & 0 \end{bmatrix}.$$

Row reduce B to find:

$$\text{RREF}(B) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The equation $B\mathbf{x} = 0$ simplifies to $x_1 - x_2 = 0$. Let $x_2 = t$, then $x_1 = t$ and $x_3 = 0$. The solution is:

$$\mathbf{x} = t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

The null basis is:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Conclusion

Finding the basis of a matrix involves identifying linearly independent vectors for a subspace (column space, row space, or null space). Each subspace is spanned by a unique set of vectors that form its basis.

Say, in a different example for Null space, we get RREF as

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

$$\Rightarrow x_1 = -2x_2 - 3x_3 - 4x_4$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 - 4x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

let $x_2 = t$
$x_3 = s$
$x_4 = r$

$$= \begin{bmatrix} -2t - 3s - 4r \\ t \\ s \\ r \end{bmatrix} \Rightarrow x = t\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis.